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Stability of Magnetic Bearing-Rotor Systems and the Effects of Gravity and Damping

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New and general stability criteria are developed for magnetic bearing-rotor systems under practical conditions of system operation and failure based on the Lyapunov second (direct) method. The unperturbed (fully nonlinear) stability of a conventional magnetic bearing-rotor configuration is analyzed for zero gravity; these results are shown to apply with gravity, based on observed similarities of the nonlinear Lagrangian equations. In addition to known results for stability and instability, the system can be stable when a magnetic bearing fails (has negative stiffness), but the net stiffness is still positive. A complete set of sufficient conditions are derived. This temporary stability depends upon inherent gyroscopic forces and is lost when dissipative forces are introduced. However, even with damping the gyroscopic forces improve the system's relative stability. The results are applicable to other gyroscopic systems.

Nomenclature

A = shaft transverse moment of inertia
C = shaft axial moment of inertia
c_{ij} = damping coefficient, $i, j = 1, 2, \ldots$
D_i = ith principal minor determinant of square matrix, $i = 1, 2, \ldots$
F = Rayleigh damping function
f_1, f_2 = stability functions, Eq. (12)
H = Hamiltonian function, $H = T + V$
k_i = magnetic bearing stiffness, $i = 1, 2$
L = Lagrangian function, $L = T - V$
l_i = distance (absolute) between magnetic bearing and reference origin, $i = 1, 2$
n = dynamic parameter, Eq. (4)
q_i = dummy variable (Hessian matrix), $i = 1, 2, \ldots$
T = kinetic energy of system
U = Lyapunov function
V = potential energy of system
α = lumped parameter, Eq. (18)
β = computational parameter
γ = lumped parameter, Eq. (18)
θ_i = rotational generalized coordinates, $i = 1, 2, 3$
λ = eigenvariable
ξ = dummy linear dynamic variables, Eq. (32)
Ω = angular momentum parameter, Eqs. (4), (17)

Introduction

MAGNETIC bearings are uniquely able to isolate a rotor system from disturbances and to suppress vibrations. Each magnetic bearing is controlled by a feedback control system which varies the shaft supporting force to minimize the vibration of the rotor system. Thus, magnetic bearings are well-suited to space station rotor systems which require a gravity-free and disturbance-free environment, for example, precision manufacturing, biological/chemical experiments, etc. However, a magnetic bearing can fail dynamically, e.g., when a control loop fails or is open-looped. Then the bearing stiffness becomes negative, such that the magnetic bearing will attract the shaft rather than repel it back to the operating point, and the system can become unstable.[1]

A typical configuration is analyzed here, per Fig. 1: there are magnetic bearings at each end of the rotating shaft and a magnetic or traditional thrust bearing at one end of the shaft. There may be conventional (passive) backup bearings adjacent to the magnetic bearings which support the shaft at rest, during startup/shutdown, and in case of magnetic bearing failure. These are loose and do not affect this analysis. (In a companion paper, a design with a central backup bearing is considered.[2] The results are quite different, and this configuration has certain advantages in terms of fail-safe stability.)

The zero-gravity applications are of principal interest and are analyzed first; the effect of gravity is then considered. New general stability criteria are derived for various modes of system failure based on the Lyapunov second (direct) method. This method has its problems, but it addresses the fully nonlinear problem without solving the equations of motion (and the restriction to specific numerical values), or the unboundedness of linearization assumptions. This method also facilitates consideration of certain nonmodeled factors on stability, e.g., dissipative forces. Some of the idiosyncracies of the Lyapunov direct method also appear, for example, the choice of variables can affect the suitability of this method.

The zero-gravity results can be applied to the case with gravity, albeit by an indirect approach. With gravity the Lyapunov method is problematic; however, the characteristic equation of the linearized Lagrangian equations of motion can be identified with the zero-gravity Lyapunov criteria, such that the latter can be applied in full. The advantage is that the Lyapunov results are very general and give results for the gravity case which are not apparent from the linearized equations of motion. Obviously this is useful, since there may be

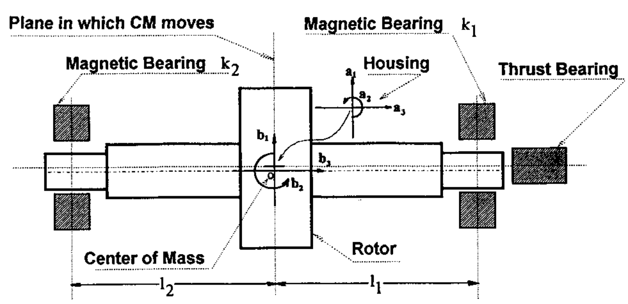

Fig. 1 Magnetic bearing system.

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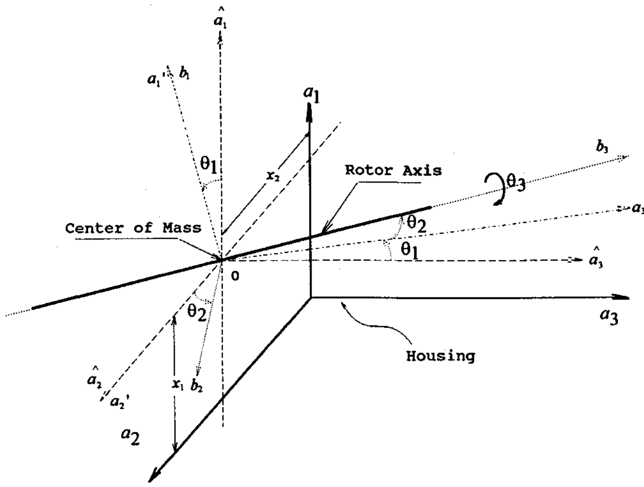


Fig. 2 System variables for Lyapunov analysis.

gravitational effects in near-Earth orbits, and magnetic bearings are widely applied in terrestrial rotor systems.

General Model

In Figs. 1 and 2, the bearing axis in the housing frame is along the $\boldsymbol{a}_3$ axis. The shaft B (of mass M) rotates about its longitudinal principal axis ($\boldsymbol{b}_3$) and is assumed to be symmetric about this axis (this is not essential). The reference point 0 is the center of mass of the shaft. Accordingly, the symmetry axes $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$, are principal axes: A, A, and C are the respective principal moments of inertia. The thrust bearing inhibits longitudinal motion of the rotor, so the shaft moves freely only in the magnetic bearing plane ($\boldsymbol{a}_1$-$\boldsymbol{a}_2$ plane); the coordinates of the center of mass are x_1 and x_2, respectively, in this frame. The bearings have stiffnesses k_1 and k_2 and are located at distances l_1 and l_2, respectively, from 0. In the following analyses k_1, k_2 and l_1, l_2 are all normally positive.

The shaft orientation variables in Fig. 2 are: θ_1 about the housing $\boldsymbol{a}_2$ axis, θ_2 about $\boldsymbol{a}_1'$ (the moved position of $\boldsymbol{a}_1$), and θ_3 about the $\boldsymbol{a}_3''$-$\boldsymbol{b}_3$ axis. (See Appendix for transformation.) The angular velocity of the rotor B with respect to the housing is

$$\boldsymbol{\omega} = \dot{\theta}_2 \boldsymbol{b}_1 + \dot{\theta}_1 \cos\theta_2 \boldsymbol{b}_2 + (\dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2)\boldsymbol{b}_3 \qquad (1)$$

The effect of a steady rotation of the entire system (an orbit relative to a true inertial frame) does not materially affect the results; this merely adds to the components of the angular velocity and would be a negligibly small effect. (However, the choice of orientation variables can influence the suitability of the Lyapunov method. See Discussion section.)

The system kinetic energy is

$$T = \tfrac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) + \tfrac{1}{2}A(\dot{\theta}_2^2 + \theta_1^2 \cos^2\theta_2) + \tfrac{1}{2}C(\dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2)^2 \qquad (2)$$

The potential energy is comprised of the magnetic bearing springs and gravity:

$$V = \tfrac{1}{2}k_1((x_1 + l_1\cos\theta_2 \sin\theta_1)^2 + (x_2 - l_1 \sin\theta_2)^2) + \tfrac{1}{2}k_2((x_1 - l_2 \cos\theta_2 \sin\theta_1)^2 + (x_2 + l_2\sin\theta_2)^2) + Mgx_1 \qquad (3)$$

From the Lagrangian, $L = T - V$, θ_3 is an ignorable (cyclic) coordinate[3–5]; hence

$$\frac{\partial L}{\partial \dot{\theta}_3} = C(\dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2) = Cn = \text{constant}, \Omega$$
$$\text{i.e., } n = \dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2 = \text{constant} \qquad (4)$$

Cn is the angular momentum about the shaft longitudinal axis $\boldsymbol{b}_3$ and is conserved.

The system described by Eqs. (2–4) is conservative, so the Hamiltonian (obtained from the Routh function) equals the total energy of the system and is constant[3–5]:

$$H = \tfrac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) + \tfrac{1}{2}A(\dot{\theta}_2^2 + \dot{\theta}_1^2\cos^2\theta_2) + \tfrac{1}{2}Cn^2 + \tfrac{1}{2}k_1((x_1 + l_1\cos\theta_2\sin\theta_1)^2 + (x_2 - l_1\sin\theta_2)^2) + \tfrac{1}{2}k_2((x_1 - l_2\cos\theta_2\sin\theta_1)^2 + (x_2 + l_2\sin\theta_2)^2) + Mgx_1 \qquad (5)$$

Stability with Zero Gravity (Space)

In space applications, gravity forces are assumed to be negligible, whence $g \approx 0$. This assumption also applies in a gravitational field when the shaft is vertical such that the gravitational force is absorbed by the thrust bearing.

A common choice of Lyapunov function is comprised of the Hamiltonian function plus any other motion integrals available; without the latter, possible stability conditions may be missed.[5–7] The bearing forces go through and are perpendicular to the $\boldsymbol{a}_3$ axis (see Fig. 2); with zero gravity, there is no moment about the $\boldsymbol{a}_3$ axis, so the angular momentum of the shaft about $\boldsymbol{a}_3$ is conserved:

$$M(\dot{x}_2 x_1 - \dot{x}_1 x_2) - A\dot{\theta}_2 \sin\theta_1 + (Cn + A\dot{\theta}_1\sin\theta_2)\cos\theta_1\cos\theta_2 = \text{constant} \qquad (6)$$

(This angular momentum component is not conserved if $g \neq 0$, since under any lateral displacement of the center of mass the gravitational force will exert a moment about the $\boldsymbol{a}_3$ axis.) Hence, from Eqs. (4–6) with $g = 0$, a Lyapunov function is:

$$U = \tfrac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) + \tfrac{1}{2}A(\dot{\theta}_2^2 + \dot{\theta}_1^2\cos^2\theta_2) + \tfrac{1}{2}Cn^2 + \tfrac{1}{2}k_1((x_1 + l_1\cos\theta_2\sin\theta_1)^2 + (x_2 - l_1\sin\theta_2)^2) + \tfrac{1}{2}k_2((x_1 - l_2\cos\theta_2\sin\theta_1)^2 + (x_2 + l_2\sin\theta_2)^2) + \beta\{M(\dot{x}_2 x_1 - \dot{x}_1 x_2) - A\dot{\theta}_2\sin\theta_1 + (Cn + A\dot{\theta}_1\sin\theta_2)\cos\theta_1\cos\theta_2\} \qquad (7)$$

β is a computational parameter, to be determined as needed.

For this system U = constant. Therefore, $\dot{U} = 0$. For stability in the sense of Lyapunov, if U is sign-definite at an isolated equilibrium point, the equilibrium point will be stable[5,6] (e.g., Theorem 6.7.1). The equilibrium point of interest is

$$q_i,\ i = 1, 8: \dot{x}_2 = \dot{x}_1 = \dot{\theta}_2 = \dot{\theta}_1 = x_2 = x_1 = \theta_2 = \theta_1 = 0$$

Stability is evaluated by Sylvester's criterion[5,6]; the Hessian matrix at this equilibrium point is

$$\left[\frac{\partial^2 U}{\partial q_i \partial q_j}\right] = \begin{bmatrix} M & 0 & 0 & 0 & 0 & \beta M & 0 & 0 \\ 0 & M & 0 & 0 & -\beta M & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 & 0 & -\beta A \\ 0 & 0 & 0 & A & 0 & 0 & \beta A & 0 \\ 0 & -\beta M & 0 & 0 & k_1 + k_2 & 0 & k_2 l_2 - k_1 l_1 & 0 \\ \beta M & 0 & 0 & 0 & 0 & k_1 + k_2 & 0 & k_1 l_1 - k_2 l_2 \\ 0 & 0 & 0 & \beta A & k_2 l_2 - k_1 l_1 & 0 & K - \beta Cn & 0 \\ 0 & 0 & -\beta A & 0 & 0 & k_1 l_1 - k_2 l_2 & 0 & K - \beta Cn \end{bmatrix} \quad (8)$$

where

$$K = k_1 l_1^2 + k_2 l_2^2 \quad (9)$$

K is the effective bearing stiffness resisting shaft rotation about an axis normal to the bearing axis $\boldsymbol{a}_3$.

For a stable system, the principal minor determinants of Eq. (8) must be sign-definite (positive):

$$\begin{aligned} D_1 &= M > 0 \\ D_2 &= M^2 > 0 \\ D_3 &= AM^2 > 0 \\ D_4 &= A^2M^2 > 0 \\ \rightarrow D_5 &= M^2A^2(k_1 + k_2 - \beta^2 M) > 0 \\ D_6 &= M^2A^2(\beta^2 M - k_1 - k_2)^2 > 0 \\ \rightarrow D_7 &= A^2M^2(k_1 + k_2 - \beta^2 M)\{(K - \beta Cn - \beta^2 A) \\ &\quad \times (k_1 + k_2 - \beta^2 M) - (k_1 l_1 - k_2 l_2)^2\} > 0 \\ D_8 &= A^2M^2\{(k_1 + k_2 - \beta^2 M)((K - \beta Cn - \beta^2 A) \\ &\quad \times (k_1 + k_2 - \beta^2 M) - (k_1 l_1 - k_2 l_2)^2)\}^2 > 0 \end{aligned} \quad (10)$$

Only D_5 and D_7 need to be investigated; the others are inherently satisfied for all values of the system parameters and β real.

$D_5 > 0$ requires that

$$k_1 + k_2 > \beta^2 M \geq 0 \quad (11)$$

This is least restrictive on the stiffnesses at $\beta = 0$.

For $D_7 > 0$, due to Eq. (11), only the bracketted { } term needs to be investigated for positivity. This is written as

$$k_1 l_1^2 + k_2 l_2^2 - \beta Cn - \beta^2 A > \frac{(k_1 l_1 - k_2 l_2)^2}{(k_1 + k_2 - M\beta^2)} \geq 0$$

i.e.,

$$f_1(\beta) > f_2(\beta) \geq 0 \quad (12)$$

That is, for stability, there must be values of β (real, including zero) for which f_1 and f_2 intersect and Eq. (11) is satisfied, as shown in Figs. 3 and 4. These intersections are established by solving Eqs. (11) and (12) for real-valued β, as functions of the bearing stiffnesses k_1 and k_2, in terms of feasible values of the other physical and operating parameters.

There are three cases to consider, two of which can be evaluated immediately:

1) $k_1 > 0$ and $k_2 > 0$.
2) The case in which $k_1 < 0$ and $k_2 < 0$. Both bearings fail.
3) The case in which k_1 or $k_2 \leq 0$. Only one bearing fails.

For case 1, all of the determinants in Eqs. (10) are positive, and Eqs. (11) and (12) are always satisfied, per Fig. 3. It is easily verified that Eq. (12) is satisfied for $\beta = 0$, since

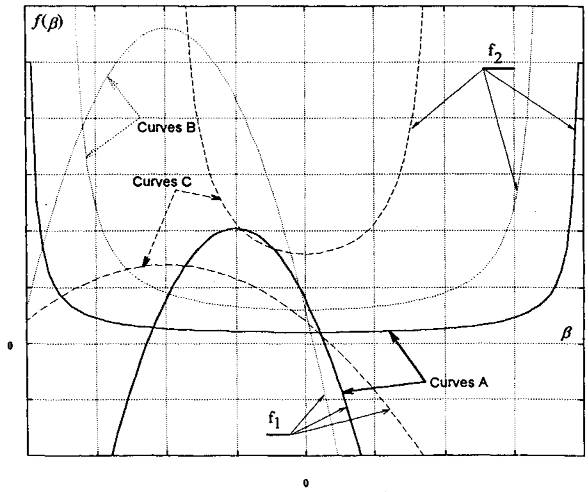
$$f_1(0) - f_2(0) = \frac{k_1 k_2 (l_1 + l_2)^2}{k_1 + k_2} > 0 \quad (13)$$

This is positive up to the limiting case of barely positive stiffness(es). However, with low stiffness(es), while the system is theoretically stable, its performance may not be satisfactory; this is a different issue. In terms of the Lyapunov criterion, U is constant and $\dot{U} = 0$, the specified equilibrium point is stable in the sense of Lyapunov—a steady oscillation. Since β can be zero, this case is independent of the angular momentum about $\boldsymbol{a}_3$. This is a well-known condition.

In case 2, both bearings' stiffnesses are negative (and presumably failed), and Eq. (11) is never satisfied: there is no feasible region, even though Eq. (12) can be satisfied. The potential energy for this condition is negative definite, so the system potential energy is maximum at this equilibrium point—an unstable equilibrium point.

Cases 1 and 2 are well-known. Both are unaffected by conserving angular momentum about $\boldsymbol{a}_3$, so $\beta = 0$ is acceptable.

Case 3 indicates that one bearing may fail (say k_1), and the system can still be stable if the magnitude of its (negative) stiffness in the failed mode is less than the unfailed bearing positive stiffness. Equation (11) requires that the net stiffness be positive, pending the value of β. For stability, the solution to Eq. (12) must yield a feasible region in terms of the system parameters for β nonzero and real, and which also satisfies Eq. (11). Then in Eq. (12), $f_2(\beta)$ is positive and possibly large; as shown hereafter, stability may be achieved by a very high rotor speed, i.e., by the inherent gyroscopic forces about $\boldsymbol{b}_3$ and $\boldsymbol{a}_3$. Unfortunately, Eq. (12) is very messy to solve analytically (even by Macsyma) to determine stable regions, and the result would be virtually useless due to its complexity. Numerical solutions are useful for specific cases, but have limited generalizeability. However, useful analytical results can be obtained, as follows.



Curves A: $k_1 > 0, k_2 > 0$. System stable.
Curves B: $k_1 = 0, k_2 > 0$. System stable.
Curves C: $k_1 < 0, k_2 > 0$. System unstable.

f_1: $k_1 l_1^2 + k_2 l_2^2 - Cn\beta - A\beta^2$

f_2: $\dfrac{(k_1 l_1 - k_2 l_2)^2}{k_1 + k_2 - M\beta^2}$

Fig. 3 General conditions for stability.

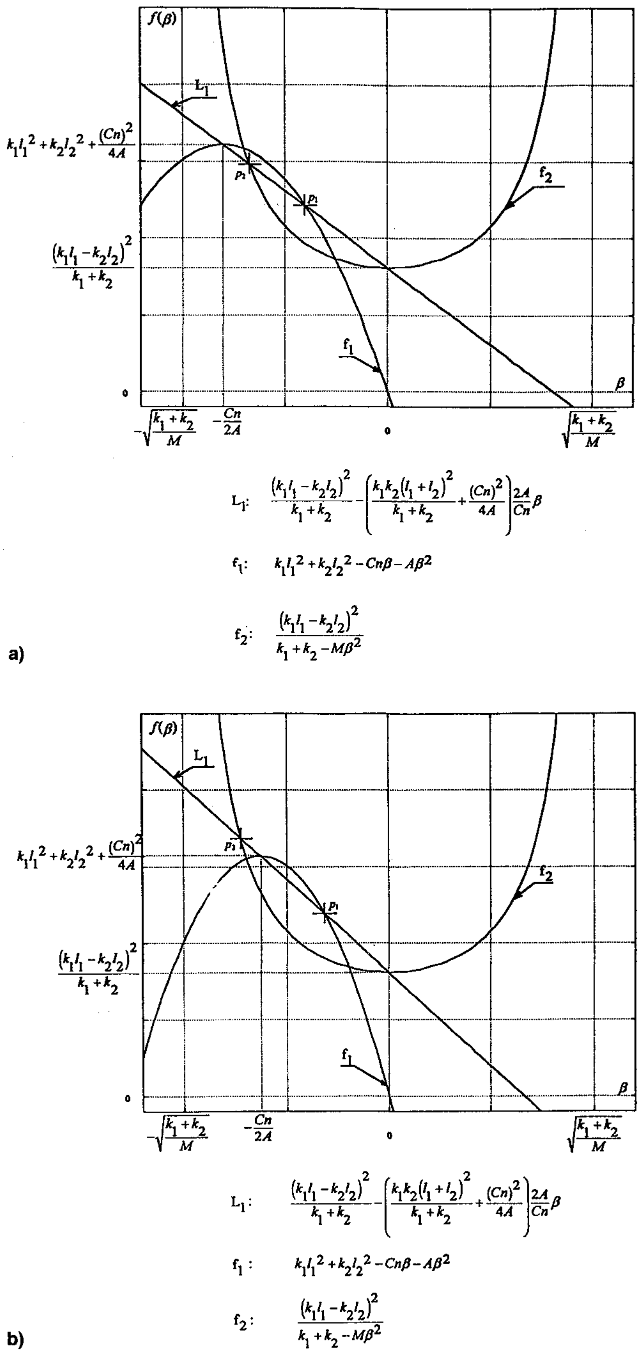


Fig. 4 Zero gravity, case 3: conditions for stability.

Case 3: Conditions for Stability

In the following discussions, k_1 will be considered as the variable stiffness (≤ 0) of the failed bearing. Here k_2 is positive and large. Figure 4 illustrates the conditions discussed below.

By Eq. (11), $f_2(\beta) > 0$, so from Eq. (12),

$$f_1(\beta) = k_1l_1^2 + k_2l_2^2 - \beta Cn - \beta^2 A > 0 \tag{14}$$

Since A is positive and β is real, this requires that

$$(Cn)^2 > -4A(k_1l_1^2 + k_2l_2^2) \tag{15}$$

β must be between the intersections of $f_1(\beta)$ and the $f(\beta) = 0$ axis. Also, $f_1(0) < f_2(0)$ because $k_1 < 0$. See Figs. 4a and 4b.

A stronger condition is obtained by observing that $f_{1(\max)}$ must be greater than $f_{2(\min)}$. These occur at $\beta = -Cn/2A$ and 0, respectively. Substitution of these values into Eq. (12) gives

$$f_1\left(-\frac{Cn}{2A}\right) = k_1l_1^2 + k_2l_2^2 + \frac{(Cn)^2}{4A}$$

$$> \frac{(k_1l_1 - k_2l_2)^2}{(k_1 + k_2)} = f_2(0) > 0$$

Since $A > 0$, β is real, and l_1 and $l_2 > 0$, Eq. (15) requires that

$$(Cn)^2 > \frac{-4Ak_1k_2(l_1 + l_2)^2}{k_1 + k_2} \tag{16}$$

Equations (15) and (16) relate the same quantities and are of the same order of magnitude; they give the same result if the bearings are equally stiff and symmetrically located. For $k_1 < 0$, it is seen that Eq. (16) dominates: it specifies a higher value of $(Cn)^2$ for stability [given by Eq. (13)] and a narrower range for β [inside that specified by Eq. (14)], per Figs. 3 and 4. To emphasize that the right side of Eq. (16) is positive for $k_1 < 0$

$$(Cn)^2 > \frac{-4Ak_1k_2(l_1 + l_2)^2}{k_1 + k_2} \underline{\Delta}\Omega^2 > 0 \tag{17}$$

For $k_1 < 0$, these are necessary conditions, but neither criterion guarantees a feasible region—i.e., intervals of β which satisfy Eq. (12) in which Eq. (11) is satisfied. Equation (17) near equality may be unstable; in Fig. 4b f_2 moves up relative to f_1 such that the required intersection does not occur. The sufficient condition is a feasible solution to Eq. (12), but this is not practical or useful, as noted earlier. An equivalent condition is to find the value of β at which the tangents to f_2 and f_1 are parallel; the system can be stable if $f_2(\beta) > f_1(\beta)$. This gives another very complicated and virtually unuseable result. A sufficient condition is if $f_1(-Cn/2A) \geq f_2(-Cn/2A)$, per Fig. 4b. This yields a messy quadratic in $(Cn)^2$ which, while solvable, is too restrictive. A better sufficiency condition is the construction shown in Fig. 4b; this includes the situation shown in Fig. 4a. A line L_1 is constructed through $f_{1(\max)}$ at $\beta = -Cn/2A$ and $f_{2(\min)}$ at $\beta = 0$; then, subject to $f_{1(\max)} > f_{2(\min)}$ which gave Eq. (17), the intersection of L_1 with f_2 (at p_2) must lie left of the intersection of L_1 and f_1 (at p_1) for real values of β. For this to be the case, after much algebra,

$$(Cn)^2 > \left(\frac{\alpha + \gamma + 1 + \sqrt{(\alpha + \gamma - 1)^2 + 4\gamma}}{2\alpha}\right)\Omega^2 \tag{18}$$

where Ω^2 is from Eq. (17) and is positive, and

$$\alpha = \frac{4A^2(k_1 + k_2)}{\Omega^2 M}, \qquad \gamma = \frac{4A(k_1l_1 - k_2l_2)^2}{\Omega^2(k_1 + k_2)}$$

α and γ are both positive, since $k_1 + k_2 > \beta^2 M > 0$. Equation (18) corresponds to

$$\beta \leq -\frac{\Omega^2}{2ACn} \tag{19}$$

That is, β is left of p_1. The expression for p_2 is very complicated. By construction, p_2 is left of p_1 for Eq. (18), so Eq. (19) gives a value of β for the Lyapunov function. Equation (18) gives a higher value for $(Cn)^2$ than Eq. (17) and is a sufficient condition for stability.

Thus, the system can be stable if one magnetic bearing fails. However, without sufficiently large gyroscopic forces, the system is unstable.

Effect of Dissipative Forces

This analysis follows Chetayev[6] (also presented in Ref. 5). Dissipative forces are characterized by a Rayleigh dissipation function (which corresponds to linear damping),

$$F = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} C_{ij} q_i q_j \tag{20}$$

The q_i are as defined for Eq. (8), viz. $\dot{x}_2$, $\dot{x}_1$, $\dot{\theta}_2$, $\dot{\theta}_1$, respectively. For complete damping, F is a positive definite function. Since H is the total energy of the system, now $\mathrm{d}H/\mathrm{d}t = -2F$, which is negative definite. H is no longer constant and can be sign variant; also, the angular momentum about $\boldsymbol{a}_3$, Eq. (6), may not be conserved. Therefore, a different Lyapunov function than Eq. (7) is required; usually $U = H$ is chosen.

For cases 1 and 2 if $U = H$, the basic properties of the system for $\beta = 0$ in the previous sections are unaffected by the presence of damping: case 1 is always stable, and case 2 is always unstable, and damping does not affect this.

However, the stability in case 3 is affected by dissipative forces, as follows.

Effect of Dissipative Forces—Case 3

Without dissipation, β must be nonzero to evaluate stability. Indeed, if $U = H$, the Lyapunov and Chetayev instability theorems[5,6] indicate that case 3 is unstable: Eq. (13) < 0. Stability can be further investigated by taking as a Lyapunov function

$$U = -H - \beta \sum_{i=1}^{2} \frac{k_1 k_2}{k_1 + k_2} \theta_i \dot{\theta}_i \tag{21}$$

where $k_1 k_2/(k_1 + k_2) < 0$ since $k_1 < 0$, and Eqs. (11) and (18) are satisfied. As above, $\mathrm{d}H/\mathrm{d}t = -2F$ and is negative definite. β is chosen (positive) so that $\dot{U}$ is positive definite in a domain which contains the equilibrium point. A region can be found in which $U \geq 0$ and which contains the equilibrium point. However, the boundary of this region is defined by $U = 0$, so the equilibrium point is on this boundary. These are the conditions of the Chetayev instability theorem.[5,6] Therefore, the equilibrium point is unstable with complete Rayleigh damping.

If some states are undamped (incomplete damping), F is positive semidefinite. The system can be stable if the nonzero damping coefficients correspond to trivial forward (in time) solutions. In this problem, all solutions are significant. So, as $\mathrm{d}H/\mathrm{d}t = -2F < 0$ for at least one nontrivial forward trajectory, the system with any dissipation is unstable by the same argument as before: the equilibrium point is not strictly inside the region, and U can change sign in this region. (This is also called pervasive damping[5].) Therefore, for incomplete/pervasive damping the system will be unstable.

To summarize, without dissipation, case 3 is orbitally stable (in the sense of Lyapunov) if Eqs. (11) and (18) are satisfied. That is, if one bearing fails, the system can be stabilized by inherent gyroscopic forces if the shaft rotation is sufficiently high. With damping, the total energy of the system is being dissipated continuously, the system slows, and the condition for stability cannot be sustained. This is called temporary stability due to gyroscopic forces, and is lost in the presence of dissipative forces.[5,6] However, the gyroscopic forces do improve the relative stability of the system.

Effect of Gravity

The orientation variables, kinetic and potential energies are as given in the previous section, except that $g > 0$. With gravity, a Lyapunov function is probably restricted to the Hamiltonian, since, in general, the angular momentum about $\boldsymbol{a}_3$ is not conserved. It can be shown (and is indicated from the foregoing) that this is not likely to be an adequate Lyapunov function. Accordingly, we first consider the corresponding Lagrangian equations of motion. The Lagrangian function is:

$$\begin{aligned} L = {} & \tfrac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) + \tfrac{1}{2}A(\dot{\theta}_2^2 + \dot{\theta}_1^2 \cos^2\theta_2) \\ & + \tfrac{1}{2}C(\dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2)^2 - \tfrac{1}{2}k_1[(x_1 + l_1 \cos\theta_2 \sin\theta_1)^2 \\ & + (x_2 - l_1 \sin\theta_2)^2] - \tfrac{1}{2}k_2[(x_1 - l_2 \cos\theta_2 \sin\theta_1)^2 \\ & + (x_2 + l_2 \sin\theta_2)^2] - Mgx_1 \end{aligned} \tag{22}$$

As before, θ_3 is a cyclic (ignorable) coordinate, repeated here:

$$\frac{\partial L}{\partial \dot{\theta}_3} = C(\dot{\theta}_3 - \dot{\theta}_1 \sin\theta_2) = Cn = \text{constant}, \ \Omega \tag{23}$$

The other equations of motion are

$$\begin{aligned} & A\ddot{\theta}_2 + \tfrac{1}{2}A\dot{\theta}_1^2 \sin 2\theta_2 + Cn\dot{\theta}_1 \cos\theta_2 + \tfrac{1}{2}K \sin 2\theta_2 \cos^2\theta_1 \\ & \quad - (k_1 l_1 - k_2 l_2)(x_1 \sin\theta_1 \sin\theta_2 + x_2 \cos\theta_2) = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} & A\ddot{\theta}_1 \cos^2\theta_2 - A\dot{\theta}_1 \sin 2\theta_2 \dot{\theta}_2 - Cn\dot{\theta}_2 \cos\theta_2 \\ & \quad + \tfrac{1}{2}K \cos^2\theta_2 \sin 2\theta_1 + (k_1 l_1 - k_2 l_2)x_1 \cos\theta_2 \cos\theta_1 = 0 \end{aligned} \tag{25}$$

$$M\ddot{x}_1 + (k_1 + k_2)x_1 + (k_1 l_1 - k_2 l_2)\cos\theta_2 \sin\theta_1 + Mg = 0 \tag{26}$$

$$M\ddot{x}_2 + (k_1 + k_2)x_2 - (k_1 l_1 - k_2 l_2)\sin\theta_2 = 0 \tag{27}$$

K is the composite stiffness defined in Eq. (9).

An isolated equilibrium point (denoted by *) for Eqs. (22–27) is defined by $\ddot{\theta}_1 = \ddot{\theta}_2 = \ddot{x}_2 = \ddot{x}_1 = 0$. In addition, physically θ_1^* and θ_2^* will be small. The following values immediately follow:

$$\dot{x}_1 = \dot{x}_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0 \tag{28}$$

However, the remaining terms of Eqs. (24–27) are not in general satisfied by the apparent point

$$\theta_1 = \theta_2 = x_2 = 0; \quad x_1 = -\frac{Mg}{k_1 + k_2} \tag{29}$$

This is only for a statically symmetric design, i.e., $k_1 l_1 = k_2 l_2$. The general equilibrium for Eqs. (24) and (27) has

$$x_2^* = 0, \quad \theta_2^* = 0 \tag{30}$$

Eqs. (25) and (26) are solved simultaneously to yield

$$\begin{aligned} x_1^* &= -\frac{MgK}{k_1 k_2 (l_1 + l_2)^2} \\ \sin\theta_1^* &= -\frac{(k_1 l_1 - k_2 l_2)x_1}{K} = \frac{(k_1 l_1 - k_2 l_2)Mg}{k_1 k_2 (l_1 + l_2)^2} \end{aligned} \tag{31}$$

The linearized equations with respect to this point are, in matrix form

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix} \xi + \begin{bmatrix} 0 & -Cn & 0 & 0 \\ Cn & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} K^* & 0 & (k_1 l_1 - k_2 l_2)c_1^* & 0 \\ 0 & K & 0 & -(k_1 l_1 - k_2 l_2) \\ (k_1 l_1 - k_2 l_2)c_1^* & 0 & (k_1 + k_2) & 0 \\ 0 & -(k_1 l_1 - k_2 l_2) & 0 & (k_1 + k_2) \end{bmatrix} \xi = 0 \tag{32}$$

where $c_1^* = \cos\theta_1^*$ per Eq. (31) and $K^* = K(\cos\theta_1^*)^2$; K is from Eq. (9). The perturbation variables, $\xi = (\theta_1', \theta_2', x_1', x_2')$, correspond to q_4, q_3, q_2, q_1, respectively, in the previous section.

The characteristic equation is

$$\begin{vmatrix} A\lambda^2 + K^* & -Cn\lambda & (k_1l_1 - k_2l_2)c_1^* & 0 \\ Cn\lambda & A\lambda^2 + K & 0 & -(k_1l_1 - k_2l_2) \\ (k_1l_1 - k_2l_2)c_1^* & 0 & M\lambda^2 + (k_1 + k_2) & 0 \\ 0 & -(k_1l_1 - k_2l_2) & 0 & M\lambda^2 + (k_1 + k_2) \end{vmatrix}$$
$$= \begin{matrix} [(A\lambda^2 + K^*)(M\lambda^2 + k_1 + k_2) - (k_1l_1 - k_2l_2)^2(c_1^*)^2] \\ \times[(A\lambda^2 + K)(M\lambda^2 + k_1 + k_2) - (k_1l_1 - k_2l_2)^2] + (Cn\lambda)^2(M\lambda^2 + k_1 + k_2)^2 = 0 \end{matrix} \tag{33}$$

Since $K^*K = K^2 \cos^2\theta_1^*$, and θ_1^* is small, $K^* + K \approx 2K \cos \theta_1^*$. (This is a slightly better approximation than simply taking $\cos \theta_1^* \approx 1$, whence $K^* \approx K$. The difference is small, but it retains the nonzero equilibrium point.) With these approximations, Eq. (33) can be factored:

$$\begin{aligned} &\{[(A\lambda^2 + Kc_1^*)(M\lambda^2 + k_1 + k_2) - (k_1l_1 - k_2l_2)^2c_1^*] \\ &+ iCn\lambda(M\lambda^2 + k_1 + k_2)]\}\{[(A\lambda^2 + Kc_1^*)(M\lambda^2 \\ &+ k_1 + k_2) - (k_1l_1 - k_2l_2)^2c_1^*] \\ &- iCn\lambda(M\lambda^2 + k_1 + k_2)]\} = 0 \end{aligned} \tag{34}$$

where $i = \sqrt{-1}$. The { } factors can be written as

$$\{(M\lambda^2 + k_1 + k_2)(A\lambda^2 + Kc_1^* \pm iCn\lambda) - (k_1l_1 - k_2l_2)^2c_1^*\} \tag{35}$$

To satisfy Eqs. (34) and (35), the roots of the polynomial in λ will be purely imaginary. Thus, the system will at best be oscillatorily stable: stable in the sense of Lyapunov, but marginally stable (unstable) by classical criteria. Let $\lambda = \pm i\omega$:

$$\begin{aligned} &\{(-M\omega^2 + k_1 + k_2)(-A\omega^2 + Kc_1^* \mp \omega Cn) \\ &- (k_1l_1 - k_2l_2)^2c_1^*\} \end{aligned} \tag{36}$$

(It makes no difference which pairs of signs are analyzed; the same results are obtained for all pairs.) If Eq. (36) is divided by $(-M\omega^2 + k_1 + k_2)$, we get Eq. (12) at equality in terms of $f_1(\omega)$ and $f_2(\omega)$. Thus, this is the limiting condition of the zero-gravity case for $D_7 > 0$ in Eqs. (10), (12), et seq. The only difference is that the numerical parameter β is replaced by the eigenparameter $\lambda = \pm i\omega$ here and the presence the equilibrium offset of $\cos \theta_1^*$. Further, it can be shown that $(-M\omega^2 + k_1 + k_2) > 0$.

Accordingly, the dynamical structure and properties of the system are identical to the zero-gravity case, except for the small equilibrium offset. Hence, the stability conditions with gravity are the same as the zero-gravity case, as delineated in cases 1, 2, and 3 in the previous section, and all of those results apply here also. Case 1 is stable, and case 2 is unstable, with or without damping. Case 3, where one bearing fails, exhibits temporary stability if Eqs. (11) and (18) are satisfied: without damping the system is oscillatorily stable due to the gyroscopic forces. This corresponds to Eq. (36). The addition of damping can insert coefficients c_{ij} anywhere in the damping matrix in Eq. (32) and $c_{ij}\lambda$ in Eq. (33). The variety of cases makes solutions difficult without recourse to numerical values, so the general characteristics of the system are less apparent than the Lyapunov results.

The linearized system per se can be evaluated by the criteria of Refs. 8, 9, and 10 by identifying the matrices in Eq. (32) as

$$M\ddot{\xi} + G\dot{\xi} + K\xi = 0 \tag{37}$$

respectively. Sufficient conditions for stability are: $4K - GM^{-1}G$ must be positive definite, and $GM^{-1}K - KM^{-1}G$ must be positive semidefinite. For this system, the latter is always satisfied. For $c\theta_1^* \approx 1$, the former gives Eq. (15) for D_1 and Eq. (16) for D_3. However, as was the case earlier, these criteria do not as clearly relate the general applicability of the nonlinear Lyapunov analysis. The zero-gravity Lyapunov results also give better stability bounds for case 3. The Lyapunov analysis is even more revealing when damping is introduced, for the same reasons as the previous paragraph.

Discussion

The zero-gravity Lyapunov analysis provides completely general stability information on the system, including the effect of any damping variant. However, the applicability of this method depends upon the specific Lyapunov function and the variables used (see following). The most interesting case, case 3, exhibits temporary stability with zero gravity, due to inclusion of the conserved angular momentum about $\boldsymbol{a}_3$ in the Lyapunov function. Linearization would indicate case 3, but the stability bounds would be suspect. In the linearized equations-of-motion approach, the addition of damping makes the resulting characteristic equations virtually unsolvable without recourse to numerical values. In this case, the methods of Refs. 8, 9, and 10 would work well. The extension to the case with gravity, albeit with respect to a slightly different equilibrium point, exhibits the same dynamical structure and stability properties as the zero-gravity case. In a sense, this is the best of two worlds: the more complete results and advantages of the Lyapunov method can be inferred from and supplement the linearization. Then, in addition to the well-known results, other conditions and bounds of stability are indicated, and the destabilizing action of damping is easily evaluated without recourse to numerical values. The disadvantage of the Lyapunov approach is that often a suitable function cannot be found—or it will apply only in a very limited sense. With gravity, this occurs when the equilibrium condition is not known a priori, since angular momentum about the housing $\boldsymbol{a}_3$ axis is generally not conserved: in general, the gravitational force does not intersect the $\boldsymbol{a}_3$ axis, so the angular momentum about $\boldsymbol{a}_3$ is not conserved. However, for this specific equilibrium point these conditions are essentially met.

This also shows the advantage of addressing a solvable or even simpler case (here $g = 0$) using a very general method and applying the results to a related, but possibly unsolvable, case. For the case with gravity, a perturbation Lyapunov function might be constructed, as suggested in Refs. 5 and 6, but this is unnecessary and will not indicate the sufficient criterion of Eq. (18).

The ability of the Lyapunov method to show the explicit stabilizing action of the gyroscopic forces can depend on the choice of variables. There are several sets of variables that can be used for modeling this system; for example, a common alternate set would be Eulerian polar coordinates, shown in Fig. 5: θ_1 about the fixed $\boldsymbol{a}_3$ axis, θ_2 about $\boldsymbol{a}_2'$ (the moved position of $\boldsymbol{a}_2$), and θ_3 about the $\boldsymbol{a}_3''$-$\boldsymbol{b}_3$ axis. In the zero-gravity case, two of the coordinates, θ_1 and θ_3, are cyclic (ignorable), and one of the equilibrium points is indeterminate, i.e., θ_1 can be any value from zero to infinity. In this problem, these coordinates will not exhibit the stabilizing gyroscopic forces (they will be absorbed in the conserved generalized momenta)

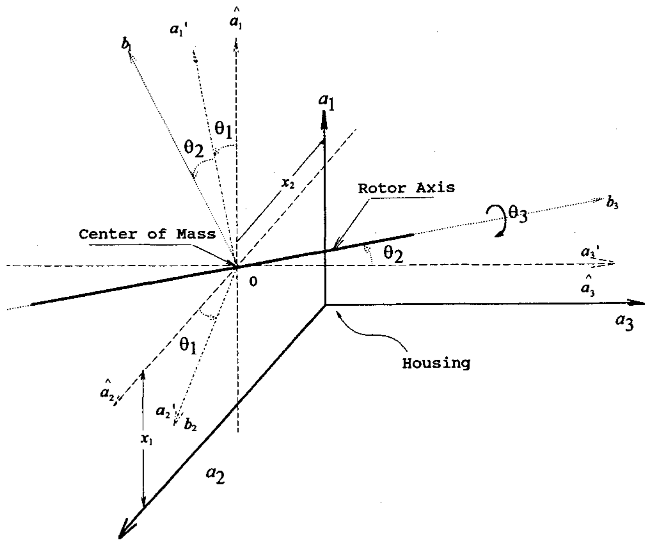


Fig. 5 Alternate orientation variables.

and will be less revealing, although they are generally more convenient for an equations-of-motion analysis. The Lyapunov method seems to work better if the equilibrium points are isolated and distinct, and preferably at the origin of the phase space. This seems to favor having the minimum number of cyclic coordinates, although the generality of these observations is not clear.

This design can be compared to a centrally supported shaft system analyzed in Ref. 2. The criteria relate similar quantities, but are quite different. Other than the always-stable case 1, there are major differences. Both configurations can be stable when one magnetic bearing fails (has negative stiffness), but the centrally supported bearing can always be stabilized. Case 2 in Ref. 2 shows that even if the failed magnetic bearing dominates the functional bearing, i.e., $|k_1| > k_2$, that system is stable. For this, case 3 is unstable here.

The temporary stability observed here in case 3 for failure of a single magnetic bearing approximately corresponds to case 3 for the centrally supported shaft system[2] where the composite stiffness K defined by Eq. (9) is the critical parameter. With the central backup bearing the dominant or even both magnetic bearings can fail, and the system can still show temporary stability. Moreover, the stabilizing speed is considerably lower. This is due to the auxiliary support point provided by the central backup bearing, about which the system can spin if a bearing fails. In the conventional design of this paper, the failed bearing cannot dominate, per Eqs. (11) and (18); if it does, the system is unstable.

To relate these designs in the context of case 3 here, consider a situation where k_2 is very large relative to k_1, which is variable. This fixes point 2 of the shaft so that the shaft effectively rotates about this point. Equation (11) is satisfied. For ease of analysis, assume Eq. (16) can be applied; then

$$(Cn)^2 > -4Ak_1 \frac{(l_1 + l_2)^2}{1 + \dfrac{k_1}{k_2}} \rightarrow -4Ak_1(l_1 + l_2)^2 >> 0$$

as $k_1 \rightarrow -k_2$

Stability is based on the value of k_1 and the rotational speed of the rotor. Essentially, the shaft is behaving like a top or gyroscope spinning about a fixed point. This is the same result as would be obtained for the centrally supported shaft with only one functioning magnetic bearing.

Conclusions

Comprehensive criteria have been developed for the stability of a conventional magnetic bearing-rotor configuration based on the unperturbed (fully nonlinear) system model. Stability was analyzed as a function of bearing stiffness (failure) under the assumptions of no gravitational effects (for any of several assumptions noted in the introduction) using the Lyapunov second (direct) method. As it turns out (by a different approach), these results apply to the case with gravity, although the equilibrium point is slightly different. The results presented here are applicable to other systems with gyroscopic effects and indicate more rigorous and complete sufficient conditions than in the literature.[8-10]

In addition to the expected results for stability and instability, the system can be stable when a magnetic bearing fails (has negative stiffness), but the net stiffness is still positive. This temporary stability depends upon inherent gyroscopic forces and is lost when dissipative forces are introduced. With magnetic bearings the shaft does not contact the bearings or other objects while in operation, so the damping forces are negligible. In the space station, an important application, ambient energy dissipation, is very small, so the conditions of temporary stability may be essentially achieved. Hence, even if a bearing fails, shaft slowing will be very gradual, and gyroscopic forces can maintain the stability of the rotor and reduce the risk of the damage of the rotor-bearing system while a shutdown or other compensation is effected.

These results were compared with a centrally supported shaft design.[2] That system exhibits temporary stability for failure of either or even both magnetic bearings, i.e., both the net stiffness and the composite stiffnesses can be negative, whereas the conventional design requires a positive net stiffness. This suggests that the centrally supported bearing is considerably more tolerant of bearing failure. This is due to the presence of the central bearing pivot about which the shaft can rotate. However, the constraints and cost of the central bearing have to be considered.

Appendix: Transformation Matrix

The shaft (B frame) orientation relative to the housing frame is established by the following sequence of finite rotations: θ_1 about the fixed $\boldsymbol{a}_2$ axis followed by θ_2 about $\boldsymbol{a}_1^*$ (the moved position of $\boldsymbol{a}_1$). These are shown in Fig. 2. The shaft spins about the $\boldsymbol{a}_3''$-$\boldsymbol{b}_3$ axis (say θ_3), which is irrelevant to the shaft orientation. Accordingly, the transformation matrix from the housing to the shaft frame is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ 0 & 1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ \sin\theta_1 \sin\theta_2 & \cos\theta_2 & \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \cos\theta_2 & -\sin\theta_2 & \cos\theta_1 \cos\theta_2 \end{bmatrix}$$

The shaft variables, given in the B frame, are expressed in the housing by the inverse (transpose) of the above transformation.

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